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THE CO-DEGREES OF IRREDUCIBLE CHARACTERS

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ABSTRACT

Let G be a finite group. The co-degree of an irreducible character χ of G is defined to be the number $|G|/\chi(1)$. The set of all prime divisors of all the co-degrees of the nonlinear irreducible characters of G is denoted by $\Sigma(G)$. First we show that $\Sigma(G) = \pi(G)$ (the set of all prime divisors of $|G|$) unless G is nilpotent-by-abelian. Then we make $\Sigma(G)$ a graph by adjoining two elements of $\Sigma(G)$ if and only if their product divides a co-degree of some nonlinear character of G. We show that the graph $\Sigma(G)$ is connected and has diameter at most 2. Additional information on the graph is given. These results are analogs to theorems obtained for the graph corresponding to the character degrees (by Manz, Staszewski, Willems and Wolf) and for the graph corresponding to the class sizes (by Bertram, Herzog and Mann). Finally, we investigate groups with some restriction on the co-degrees. Among other results we show that if G has a co-degree which is a p-power for some prime p, then the corresponding character is monomial and $O_p(G) \neq 1$. Also we describe groups in which each co-degree of a nonlinear character is divisible by at most two primes. These results generalize results of Chillag and Herzog. Other results are proved as well.

Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ the set of all ordinary irreducible characters of G. Let $\chi \in \mathrm{Irr}(G)$. The number $|G|/\chi(1)$ will be called the co-degree of χ. Co-degrees were introduced in [4] where results on groups with restrictions on the co-degrees were proved. Our purpose (which is the purpose of [4]) is to study co-degrees analogously to the recent studies by various authors of the set of character degrees and the set of sizes of the conjugacy classes.

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We start by investigating the set $\Sigma(G)$ of the relevant primes with respect to co-degrees, that is: $\Sigma(G) = \{p \mid p$ is a prime divisor of a co-degree of some nonlinear character of $G\}$. Our first result is to show that if G is not nilpotent-by-abelian then $\Sigma(G) = \pi(G)$, where $\pi(G)$ is the set of all prime divisors of $|G|$. Explicitly, we show:

THEOREM 1. *Let G be a finite nonabelian group. Set $\pi = \Sigma(G)$. Then $|\pi| < |\pi(G)|$ if and only if G' is a nilpotent π-group, $G = G'(L \times K)$, $G' \cap (L \times K) = 1$ where L is an abelian π-group, $K \neq 1$ a cyclic π' group and $G'K$ is a Frobenius group with kernel G'.*

Here X' is the commutator subgroup of the group X. A consequence of Theorem 1 is (compare this with Thompson's theorem [11], Corollary 12.2):

COROLLARY. *Let G be a finite group which is not nilpotent-by-abelian and let p be a prime divisor of $|G|$. Then G has an irreducible character with positive p-defect.*

Theorem 1 for the case $|\pi| = 1$ is proved in [4]. The character degrees analog is the Michler–Ito theorem that states that if p is a prime such that $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$, then G has a normal abelian Sylow p-subgroup.

Next we make $\Sigma(G)$ a graph as follows: connect two elements $p, q \in \Sigma(G)$, $p \neq q$, if there exists a nonlinear character $\chi \in \mathrm{Irr}(G)$, such that pq divides $|G|/\chi(1)$. Note that the graph is empty if and only if all irreducible characters are linear (i.e. G is abelian). An empty graph is considered to be connected.

The distance $d(p,q)$ of two elements $p, q \in \Sigma(G)$ is defined to be the length of the shortest path between p and q. If no path exists the distance is defined to be infinite. The diameter of a connected graph is the largest distance between two points in the graph and the diameter of a general (finite) graph is the largest diameter of the connected components of the graph.

An analog graph corresponding to character degrees was introduced in [15] and [16] where it was shown that the number of connected components of the graph is at most 3, and in the case of solvable groups the diameter of the graph is at most 3. Similar graphs corresponding to the sizes of conjugacy classes were studied in [1] where it is shown that the graph has at most 2 connected components and the diameter is at most 4. (In fact, this graph has the conjugacy classes as vertices and two classes are joined if their sizes are not relatively prime. This implies that the corresponding graph, in which the vertices are the primes occurring in all sizes of conjugacy classes and two primes are joined if both divide a size of a conjugacy class, has diameter at most 5.) Our analog is easily proved:

THEOREM 2. *Let G be a finite group. Then $\Sigma(G)$ is a connected graph whose diameter is at most* 2.

REMARK. The group $\Gamma(8)$ (see [2], p. 378) has co-degrees 56 and 24 so that $d(3,7) = 2$. Thus the diameter is 2.

We give more information on $\Sigma(G)$. In fact, $\Sigma(G)$ contains a "big" complete subgraph (consisting of all the primes dividing $|G'|$ in most cases) and every other prime is connected to this "big" subgraph. We recall that a complete graph is a graph in which $d(a,b) \leq 1$ for all vertices a,b. In order to describe a "big" subgraph we need the following notation:

For every nonlinear $\chi \in \operatorname{Irr}(G)$, we let $\rho_G(\chi) = \{p \mid p$ a prime divisor of $|G|/\chi(1)\}$. If the context allows we set $\rho_G(\chi) = \rho(\chi)$. Clearly, every subset of $\rho(\chi)$ is a complete subgraph of $\Sigma(G)$. Let π be a subset of $\pi(G)$. We say that π is a perfect subset if $\pi \subseteq \rho(\chi)$ for some nonlinear $\chi \in \operatorname{Irr}(G)$. We call π an almost-perfect subset if all elements of π, except possibly one of them, is contained in $\rho(\chi)$ for some nonlinear $\chi \in \operatorname{Irr}(G)$. We show:

THEOREM 3. *Let G be a finite nonabelian group. Then $\pi(G') \subseteq \Sigma(G)$ and the following are true*:

(a) *If $r \in \Sigma(G) - \pi(G')$ then there exists an $s \in \pi(G')$ such that $d(r,s) = 1$.*

(b) *If $G'' < G'$ then $\pi(G')$ is a perfect subset of $\Sigma(G)$. In particular, $\pi(G')$ is a complete subgraph of $\Sigma(G)$.*

(c) *If $G'' = G'$ and G'/M is a composition factor of G' then $\Sigma(G) = \pi(G)$ and one of the following holds*:

  (i) $G'/M \cong A_n$ for $n \notin \{2,3,4,5,6,8\}$ or one of the following sporadic groups: M_{12}, J_2, *HS*, J_3, *He*, *Ru*, *Suz*, Co_3, *HN*, Co_1. In this case $\pi(G')$ is a perfect subset of $\Sigma(G)$. In particular, $\pi(G')$ is a complete subgraph of $\Sigma(G)$.

  (ii) $G'/M \cong A_5, A_6, A_8$ or one of the following sporadic groups: M_{11}, J_1, M_{22}, M_{23}, M_{24}, *Mcl*, *O'N*, Co_2, F_{22}, *Ly*, *Th*, F_{23}, F_2. In this case $\pi(G')$ is a complete subgraph of $\Sigma(G)$ and an almost-perfect subset of $\Sigma(G)$.

  (iii) G'/M is a Chevalley group of characteristic p. In this case $\pi(G') - \{p\}$ is a perfect subset of $\Sigma(G)$. In particular $\pi(G') - \{p\}$ is a complete subgraph of $\Sigma(G)$.

  (iv) G'/M is one of the sporadic groups J_4, F'_{24}, F_1. In this case $\pi(G')$ is a complete subgraph of $\Sigma(G)$.

Part (iv) refers to 3 sporadic groups; more details on the graph for this case can be found in a remark after the proof of Theorem 3.

The proof for the case $G' > G''$ (in particular, for solvable groups) does not use the classification of the finite simple groups while the case $G' = G''$ does use it.

Theorems 1, 2 and 3 are proved in Section I. Theorem 2 will follow from the fact that no two co-degrees are relatively prime. In Section II we consider groups with some restrictions on the co-degrees. In [4] it is shown that if a finite group has a prime power co-degree, then G is not a nonabelian simple group. We show that:

THEOREM 4. *Let G be a finite group. Assume that there exists an irreducible character χ of G, such that $|G|/\chi(1) = p^a$ where p is a prime. Then χ is monomial and the generalized Fitting subgroup of G is a p-group, that is $F^*(G) = O_p(G) \neq 1$.*

Theorem 4 follows from a more general theorem (see Theorem 2.1); its proof depends on the classification of the finite simple groups.

THEOREM 5. *Let G be a finite nonabelian group. Assume that for each nonlinear irreducible character χ, of G, the number $|G|/\chi(1)$ has at most two distinct prime divisors. Then one of the following holds:*

(1) *G is isomorphic to one of the following nonsolvable groups:* $A_5 \cong$ PSL(2,5), PSL(2,7), $A_6 \cong$ PSL(2,9), PSL(2,8), PSL(2,17), S_6, M_{10}, PΓL(2,8). (*M_{10} is the one point stabilizer in M_{11}.*)

(2) *G is solvable and satisfies one of the following:*

- (a) *$|G|$ is divisible by at most two primes.*
- (b) *G' is a nilpotent $\{p,r\}$-group for some primes p and r and a nontrivial Hall $\{p,r\}'$-subgroup of G acts fixed-point-freely on G'. Also $\Sigma(G) = \{p,r\}$.*
- (c) *$F(G) = O_p(G)$ for some prime p, $G/F(G)$ is nonabelian, $(G/F(G))'$ is q-group for some prime q and $|\pi(G)| = 3$.*
- (d) *$F(G) = O_p(G)$ for some prime p and $G/F(G)$ is abelian.*

Note that in cases (a),(b),(c) the size of $\Sigma(G)$ is bounded. We will give an example showing that $\Sigma(G)$ is unbounded in case (d). Theorem 5 was proved for simple groups in [4]. D. Gluck ([9]) showed that if every character degree of a solvable group G has at most 2 distinct prime divisors then

$$|\{p \mid p \text{ is a prime divisor of } \chi(1) \text{ for some } \chi \in \mathrm{Irr}(G)\}| \leq 4.$$

An analog for the lengths of conjugacy classes was proved in [8]. The example mentioned above will show that no analog exists for co-degrees. This implies that the analog for co-degrees of Huppert's so-called σ–ρ conjecture (see [14], question on p. 66) is false.

Some remarks on examples and on when a co-degree of an irreducible character is a π-number for $\pi \subseteq \pi(G)$ are also included in Section II. Groups in which all co-degrees of nonlinear characters are square-free are supersolvable (see [4], Th. 3). Groups in which all co-degrees of nonlinear characters are cube-free are described in Theorem 2.3. The solvable ones have derived length no bigger than 3.

Our notation is standard and taken mainly from [11].

I. The set $\Sigma(G)$ of relevant primes

PROOF OF THEOREM 1. Assume that $|\pi| < |\pi(G)|$ and let m be the π'-part of $|G|$. Then $m \neq 1$ and, by assumption, $m \mid \chi(1)$ for all nonlinear $\chi \in \mathrm{Irr}(G)$. So, $|G| = |G:G'| + m^2 k$ for some natural number k. Thus $m \mid |G:G'|$ and G' is a π-group. Let H/G' be a π-Hall subgroup of G/G'. Then H is a normal π-subgroup of G and $G = HK$ for some abelian π'-Hall subgroup K, $|K| = m$.

Let $\theta \in \mathrm{Irr}(H) - \mathrm{Irr}(H/G')$ and $\chi \in \mathrm{Irr}(G)$ such that $(\chi_H, \theta) \neq 0$. Write $\chi_H = e(\theta_1 + \theta_2 + \cdots + \theta_t)$ where $\theta_1, \theta_2, \ldots, \theta_t$ are the distinct conjugates of $\theta_1 = \theta$. Then $\chi(1) = et\theta(1)$. Note that $\chi(1) \neq 1$ for otherwise $\chi_H = \theta$ and so $G' \subseteq \mathrm{Ker}\,\chi \cap H \subseteq \mathrm{Ker}\,\theta$, which contradicts the choice of θ. As $m \mid \chi(1)$, $\theta(1) \mid |H|$ and $(|H|, m) = 1$, we get that $m \mid et$. On the other hand

$$1 = (\chi, \chi) \geq (1/|G:H|)(\chi_H, \chi_H) = (1/|G:H|)e^2 t.$$

Thus, $e^2 t \leq |G:H| = m \leq et$. So $e = 1$ and $t = m$. This means that K acts regularly on the conjugates of θ.

Let $x \in K - \{1\}$, then the previous paragraph shows that x does not fix any element of $\mathrm{Irr}(H) - \mathrm{Irr}(H/G')$. But x centralizes H/G' so x fixes each element of $\mathrm{Irr}(H/G')$. Hence x fixes exactly $|H/G'|$ elements of $\mathrm{Irr}(H)$ and therefore it fixes exactly $|H/G'|$ conjugacy classes of H (by Brauer's Lemma). Now, x centralizes H/G' and acts coprimely on H. Consequently $H = G'C_H(x)$. So each left coset of G' in H has the form uG', where u is fixed by x. So the full H-conjugacy class of u, which clearly lies in uG', is a fixed class of x. Since x fixes exactly $|H/G'|$ H-conjugacy classes, we see that x fixes exactly one class in each coset of G' in H. As x fixes the class $\{1\} \subseteq G'$, it cannot fix any other H-class contained in G'. It follows that x, and therefore K, acts on G' without fixed points. Thus $G'K$ is a Frobenius group with Frobenius kernel G' and K is cyclic. In particular, G' is nilpotent. Let $x \in K - \{1\}$. As shown above $G' \cap C_G(x) = 1$. Observe that $G = HK = G'C_H(x)K \leq G'C_G(x)$. Therefore $C_G(x)$ is a complement of G' in G. Let $L = C_G(x) \cap H$. As $K \subseteq C_G(x)$ and $C_G(x)$ is abelian we get that $C_G(x) = L \times K$, as claimed.

Conversely, assume that G' is a nilpotent π-group, $G = G'(L \times K)$, $G' \cap (L \times K) = 1$ where L is an abelian π-group, $K \neq 1$ a cyclic π'-group and $G'K$ is a Frobenius group with kernel G'. Let $\chi \in \mathrm{Irr}(G)$, $\chi(1) \neq 1$. Write $\chi_{G'} = f(\eta_1 + \eta_2 + \cdots + \eta_s)$, where the η_i's are the distinct conjugates of $\eta = \eta_1$ in G and f is a natural number. As G' is not contained in $\mathrm{Ker}\,\chi$ we know that $\eta \neq 1_{G'}$. Let $y \in K - \{1\}$. As y acts coprimely and without fixed point on $G' - \{1\}$ we know that y does not fix any nontrivial conjugacy class of G'. It follows that y does not fix η. Let I be the inertia group of η in G. Then $K \cap I = 1$, $s = |G:I|$ and, as $G' \subseteq I$, I is normal in G. Consequently $|K| \mid s \mid \chi(1)$. Since $|K|$ is the π' part of $|G|$ we conclude that $|G|/\chi(1)$ is a π-number for all nonlinear $\chi \in \mathrm{Irr}(G)$. As $|K| \neq 1$ we have that $|\Sigma(G)| < |\pi(G)|$. This ends the proof. ∎

Theorem 2 follows from the following Proposition.

PROPOSITION 1.1. *Let G be a finite group, $\chi, \theta \in \mathrm{Irr}(G)$. Then $|G|/\chi(1)$ and $|G|/\theta(1)$ are not relatively prime.*

PROOF. Assume that $|G|/\chi(1)$ and $|G|/\theta(1)$ are relatively prime. Let $p \in \pi(G)$ and P a Sylow p-subgroup of G. Since p cannot divide both $|G|/\chi(1)$ and $|G|/\theta(1)$ we get that $|P|$ divides either $\chi(1)$ or $\theta(1)$. As this is true for all $p \in \pi(G)$, we conclude that $|G| \mid \chi(1)\theta(1)$. This is impossible as $\chi(1) < \sqrt{|G|}$ and $\theta(1) < \sqrt{|G|}$. ∎

PROOF OF THEOREM 2. Let $p, q \in \Sigma(G)$. Then there are nonlinear $\chi, \theta \in \mathrm{Irr}(G)$ such that $p \mid |G|/\chi(1)$ and $q \mid |G|/\theta(1)$. By Proposition 1.1, there is prime r which divides both $|G|/\chi(1)$ and $|G|/\theta(1)$. Thus $d(p,r) \leq 1$, $d(q,r) \leq 1$ and so $d(p,q) \leq 2$. This proves the Theorem. ∎

REMARK. Consider the graph whose vertices are the nonlinear characters and two such are connected if and only if their co-degrees are not relatively prime. Then Proposition 1.1 is just the statement that this graph is complete.

A minor adjustment to the proof of Proposition 1.1 yields:

PROPOSITION 1.2. *Let G be a finite group, $\chi, \theta \in \mathrm{Irr}(G)$. Then the greatest common divisor of $|G|/\chi(1)$ and $|G|/\theta(1)$ is at least* 3.

PROOF. Left to the reader.

LEMMA 1.3. *Let G be a finite group and N a normal subgroup of G. Set $X = N$ or G/N. Then*:

(a) $\Sigma(X) \subseteq \Sigma(G)$.

(b) *If the distance between two elements p, q of $\Sigma(X)$ is one, then the distance between p and q in $\Sigma(G)$ is also one.*

(c) *If G is nonsolvable, then $\Sigma(G) = \pi(G)$.*

PROOF. Let X be either N or G/N.

(a) Let $p \in \Sigma(X)$. Then there exists a nonlinear $\theta \in \mathrm{Irr}(X)$ such that p divides $|X|/\theta(1)$. By ([4], Lemma), $|X|/\theta(1)$ divides $|G|/\chi(1)$ for some nonlinear $\chi \in \mathrm{Irr}(G)$ so that p divides $|G|/\chi(1)$ and $p \in \Sigma(G)$.

(b) By assumption, there is a nonlinear $\theta \in \mathrm{Irr}(X)$ such that pq divides $|X|/\theta(1)$. By ([4], Lemma), $|X|/\theta(1)$ divides $|G|/\chi(1)$ for some nonlinear $\chi \in \mathrm{Irr}(G)$ so that pq divides $|G|/\chi(1)$. Thus the distance between p and q in $\Sigma(G)$ is one.

(c) This is a corollary of Theorem 1.

The proof of Theorem 3 for the case $G' > G''$ is a consequence of the next Lemma.

LEMMA 1.4. *Let G be a finite group such that $G' > G''$. Then there exists a nonlinear $\chi \in \mathrm{Irr}(G)$ such that $|G'| \mid |G|/\chi(1)$. In particular, $\pi(G')$ is a perfect subset of $\Sigma(G)$ and $\pi(G')$ is a complete subgraph of $\Sigma(G)$.*

PROOF. If all characters in $\mathrm{Irr}(G/G'')$ are linear, then $G' \subseteq \bigcap\{\mathrm{Ker}\,\theta \mid \theta \in \mathrm{Irr}(G/G'')\} = G''$. This contradicts our assumption. Hence there exists a nonlinear $\chi \in \mathrm{Irr}(G/G')$. Since G'/G'' is an abelian normal subgroup of G/G'', a theorem of Ito ([11], p. 84) implies that $\chi(1) \mid |G/G''|/|G'/G''| = |G:G'|$. Therefore, $|G'| \mid |G|/\chi(1)$ and since $\chi(1) \neq 1$ we get that $\pi(G')$ is a perfect subset of $\Sigma(G)$ and that any two elements of $\pi(G')$ are connected by an edge in $\Sigma(G)$. This proves the Lemma.

PROPOSITION 1.5. *Let G be a finite nonabelian simple group. Then:*

(1) $\Sigma(G) = \pi(G)$.

(2) *If $G \cong A_n$, $n \neq 5,6,8$ or one of the following sporadic groups: M_{12}, J_2, HS, J_3, He, Ru, Suz, Co_3, HN, Co_1, then $\pi(G)$ is a perfect subset and hence a complete graph.*

(3) *If G is a Chevalley group of characteristic p, then $\pi(G) - \{p\}$ is a perfect subset of $\Sigma(G)$, and hence a complete subgraph of $\Sigma(G)$. Moreover, there exists a $q \in \Sigma(G) - \{p\}$ such that $d(p,q) = 1$.*

(4) *If G is either A_5, A_6, A_8 or a sporadic group from the list: M_{11}, J_1, M_{22},*

M_{23}, M_{24}, Mcl, $O'N$, Co_2, F_{22}, Ly, Th, F_{23}, F_2, then $\pi(G)$ is a complete graph and an almost perfect subset.

(5) *If G is one of the sporadic groups J_4, F'_{24}, F_1, then $\pi(G)$ is a complete graph.*

PROOF. Part (1) follows from Lemma 1.3. Parts (4) and (5) and the statement on the sporadic groups in part (2) and on A_7 is obtained by checking [6]. The χ such that $\pi(G) - \{p\} \subseteq \rho(\chi)$ in part (3) is the Steinberg character; the second claim of part (3) follows from Theorem 2. Thus, let $G \cong A_n$, $n \geq 9$. By ([13], p. 139) we see that S_n possesses irreducible characters of degrees $n-1$ and $n(n-2)(n-4)/3$. Thus, A_n has characters of co-degrees $c_1 = n(n-2)!/2$ (the permutation character remains irreducible for A_n) and $c_2 = 3(n-1)(n-3) \times (n-5)!d/2$, where $d = 1$ or 2. Note that c_1 and c_2 are both even as $n \geq 9$.

Assume first that $n-1$ is not a prime and let p be an odd prime divisor of $n!/2$. If $(p, n-1) = 1$ then $p \mid c_1$. If $p \mid n-1$, then $p < n-1$ since $n-1$ is not a prime. Thus $p \mid (n-2)! \mid c_1$. Hence all prime divisors of $n!/2$ divide c_1 and we are done.

We may assume now that $n-1$ is a prime. In particular n, $n-2$ and $n-4$ are all even and $n \geq 10$. Let p be an odd prime divisor of $n!/2$. If $(p, n(n-2) \times (n-4)) = 1$ then $p \mid c_2$. If $p \mid n(n-2)(n-4)$ then either $p \mid (n/2)$ or $p \mid (n-2)/2$ or $p \mid (n-4)/2$. As $n \geq 10$ we get in all cases that $p \leq n-5$ so that $p \mid (n-5)! \mid c_2$. This proves the Lemma. ∎

LEMMA 1.6. *Let X be a finite nonidentity group such that $X' = X$. Let $G = X/M$ be a composition factor of X. Then $\Sigma(X) = \pi(X)$ and one of the following holds*:

(1) *G is isomorphic to one of the groups in Proposition* 1.5(2) *and $\pi(X)$ is a perfect subset and hence a complete graph.*

(2) *G is a Chevalley group of characteristic p and $\pi(X) - \{p\}$ is a perfect subset of $\Sigma(X)$ and hence a complete subgraph of $\Sigma(X)$.*

(3) *G is isomorphic to one of the groups in Proposition* 1.5(4) *and $\pi(X)$ is a complete graph and an almost perfect subset.*

(4) *G is isomorphic to one of the groups in Proposition* 1.5(5) *and $\pi(X)$ is a complete graph.*

PROOF. As X is nonsolvable we know that $\Sigma(X) = \pi(X)$ (Lemma 1.3). As $X' = X$, X/M is a nonabelian simple group. Let $\chi \in \mathrm{Irr}(X/M) \subseteq \mathrm{Irr}(X)$, $\chi(1) \neq 1$. Then $|X|/\chi(1) = |M||X/M|/\chi(1)$ so that $\pi(M) \subseteq \rho_X(\chi)$ and $\rho_{X/M}(\chi) \subseteq \rho_X(\chi)$.

(1) By Proposition 1.5(2) there exists a nonlinear $\chi \in \mathrm{Irr}(X/M)$ such that $\pi(X/M) \subseteq \rho_X(\chi)$. Since $\pi(M) \subseteq \rho_X(\chi)$ we get that $\pi(X) \subseteq \rho_X(\chi)$ as required.

(2) By Proposition 1.5(3) there exists a nonlinear $\chi \in \mathrm{Irr}(X/M)$ such that $\pi(X/M) - \{p\} \subseteq \rho_X(\chi)$. Since $\pi(M) \subseteq \rho_X(\chi)$ we get that $\pi(X) - \{p\} \subseteq \rho_X(\chi)$ as required.

(3) By Proposition 1.5(4) $\pi(X/M)$ is a complete graph. Let $p, q \in \pi(X)$. If both p and q are in $\pi(X/M)$ then $d(p,q) \leq 1$. If both are in $\pi(M)$ then $d(p,q) \leq 1$ as $\pi(M) \subseteq \rho_X(\chi)$ for some nonlinear $\chi \in \mathrm{Irr}(X)$. Finally if, say, $p \in \pi(M)$ and $q \notin \pi(M)$, then $q \in \pi(X/M) = \Sigma(G)$ so that $q \mid |M|\,|X/M|/\theta(1)$ for some nonlinear $\theta \in \mathrm{Irr}(X/M)$. Then $d(p,q) = 1$ and so $\pi(X)$ is a complete graph.

By Proposition 1.5(4), there exists a nonlinear $\eta \in \mathrm{Irr}(X/M)$ such that $\pi(X/M) - \{p\} \subseteq \rho_X(\eta)$ for some prime $p \in \pi(X/M)$. Since $\pi(M) \subseteq \rho_X(\eta)$ we get that $\pi(X) - \{p\} \subseteq \rho_X(\eta)$ and hence $\pi(X)$ is an almost-perfect subset.

(4) Reread the first paragraph of the proof of (3). Use Proposition 1.5(5). ■

Proof of Theorem 3. (a) Let $\chi \in \mathrm{Irr}(G) - \{1_G\}$ be such that $r \mid |G|/\chi(1)$. Let $\theta \in \mathrm{Irr}(G')$ be a constituent of $\chi_{G'}$. By ([11], Cor. 11.29, p. 190) $|G'|/\theta(1) \mid |G|/\chi(1)$. Take a prime divisor, s, of $|G'|/\theta(1)$; then $d(r,s) = 1$.

(b) This was proved in Lemma 1.4.

(c) Now G' is a perfect group, so that G' satisfies one of the four possible conclusions for X in Lemma 1.6. Let $\sigma \in \mathrm{Irr}(G')$, $\sigma(1) \neq 1$. Then $|G'|/\sigma(1) \mid |G|/\chi(1)$ for some nonlinear $\chi \in \mathrm{Irr}(G)$ (see, e.g., Lemma of [4]) and hence $\rho_{G'}(\sigma) \subseteq \rho_G(\chi)$. It follows that a complete subgraph of $\Sigma(G') = \pi(G')$ is a complete subgraph of $\Sigma(G) = \pi(G)$. Also, a perfect (respectively, almost-perfect) subset of $\Sigma(G')$ is a perfect (respectively, almost-perfect) subset of $\Sigma(G)$. Now each conclusion of part (c) follows from the corresponding one in Lemma 1.6. ■

Remark. Theorem 3, part (iv), refers to 3 sporadic groups J_4, F_{24}' and F_1. If $G'/M \cong J_4$ then there exists $\chi, \theta \in \mathrm{Irr}(G) - \{1_G\}$ such that $\pi(G') \subseteq \rho_G(\chi) \cup \rho_G(\theta)$ and $\pi(G') \subseteq \rho_G(\chi) - \{31, 43\}$. If $G'/M \cong F_{24}'$ then there exist $\chi, \theta, \eta \in \mathrm{Irr}(G) - \{1_G\}$ such that $\pi(G') \subseteq \rho_G(\chi) \cup \rho_G(\theta) \cup \rho_G(\eta)$ and $\pi(G') \subseteq \rho_G(\chi) - \{17, 23\}$. Finally, if $G'/M \cong F_1$ then there exist $\chi, \theta \in \mathrm{Irr}(G) - \{1_G\}$ such that $\pi(G') \subseteq \rho_G(\chi) \cup \rho_G(\theta)$ and $\pi(G') \subseteq \rho_G(\chi) - \{47, 59, 71\}$.

II. Restrictions on the co-degrees

Theorem 4 will be a consequence of the next theorem.

Theorem 2.1. *Let Λ be a set of natural numbers satisfying the following conditions*:

(1) *If* $n \in \Lambda$ *and* $m \mid n$ *then* $m \in \Lambda$.

(2) *If* $n \in \Lambda$ *and* n *divides the order of a direct product* S *of isomorphic nonabelian simple groups, then* $n^2 < |S|$.

Let $\pi = \pi(\Lambda)$ *be the set of primes in* Λ *and* G *a finite group. Assume that* G *has an irreducible character* χ, *such that* $|G|/\chi(1) \in \Lambda$. *Then* χ *is induced from a solvable* π*-subgroup and* $F^*(G) = F(G)$ *is a* π*-group.*

Proof. Note that if $k \in \Lambda$ then k is a π-number.

Write $\chi = \eta^G$, where η is a primitive character of some subgroup H of G (possibly, $G = H$). Then $|H|/\eta(1) = |G|/\chi(1) \in \Lambda$. Let M/N be a chief factor of H and let ϕ and τ be irreducible constituents of η_M and η_N, respectively. Then $\eta(1) = k\phi(1)$ where k is a natural number dividing $|H:M|$. Thus $|H|/\eta(1) = (|H:M|/k)(|M|/\phi(1))$ and by assumption (1) on Λ we get that $|M|/\phi(1) \in \Lambda$.

Since η is primitive, η_M and η_N are homogeneous and therefore $\phi_N = t\tau$ for some natural number t. By ([7], 51.7) there is an irreducible projective character ζ of M and an irreducible projective character ϑ of M/N such that $\phi = \zeta \otimes \vartheta$ and the degree of ζ is equal to $\tau(1)$. Then $|M|/\phi(1) = (|N|/\zeta(1))(|M:N|/\vartheta(1))$. It follows that $|M:N|/\vartheta(1) \in \Lambda$. Since $\vartheta(1) \leq \sqrt{|M:N|}$ we get that $l = |M:N|/\vartheta(1) \geq \sqrt{|M:N|}$. Since $l \in \Lambda$, it follows from property (2) of Λ that M/N is not a direct product of isomorphic nonabelian simple groups. The above then shows that $|M:N|_\pi \geq l > 1$ so that M/N, which is an arbitrary chief factor of H, is a p-group for some prime $p \in \pi$. Thus H is a solvable π-group and χ is induced from H.

Now, $F^*(G) = L(G)F(G)$ where $F(G)$ is the Fitting subgroup of G and $L(G)$ is the layer of G, that is, $L(G)$ is a central product of components. Assume that there exists a nontrivial component T, i.e. T is a perfect subnormal subgroup and $T/Z(T)$ is a nonabelian simple group. By repeated application of Clifford's theorem,

$$|G|/\chi(1) = (|G:L(G)|/n)(|L(G):T|/m)(|T|/\eta(1)) \qquad \text{for some } \eta \in \mathrm{Irr}(T),$$

n a divisor of $|G:L(G)|$ and m a divisor of $|L(G):T|$. It follows that $|T|/\eta(1) \in \Lambda$. Since $\eta_{Z(T)} = s\gamma$ for some $\gamma \in \mathrm{Irr}(Z(T))$ and s a natural number, we again can write $\eta = \alpha \otimes \beta$, where α is a projective irreducible character of T and β an irreducible projective character of $T/Z(T)$ such that $\alpha(1) = \gamma(1)$. Now, $|T|/\eta(1) = (|Z(T)|/\alpha(1))(|T/Z(T)|/\beta(1))$ and consequently $|T/Z(T)|/\beta(1) \in \Lambda$, by property (1) of Λ. Since $\beta(1) \leq \sqrt{|T:Z(T)|}$ it again follows that $|T\backslash Z(T)|/\beta(1) \geq \sqrt{|T:Z(T)|}$ which contradicts property (2) of Λ. Hence no such T exists and $L(G) = 1$. Now $F^*(G) = F(G)$. If p is a prime divisor of $F(G)$, then $K =$

$O_p(G) \neq 1$ and $|G|/\chi(1) = (|G:K|/d)(|K|/\rho(1))$ for some $\rho \in \mathrm{Irr}(K)$ and d a divisor of $|G:K|$. Thus $p \mid |G|/\chi(1)$ and so $p \in \pi$. We conclude that $F(G)$ is a π-group, as claimed. ∎

PROOF OF THEOREM 4. Let Λ be the set of all the powers of p. Note that $\pi(\Lambda) = \{p\}$. By ([12], 3.6) if P is a Sylow p-subgroup of the finite nonabelian simple group S, then $|P|^2 < |S|$. It follows that Λ satisfies the assumptions on Λ of Theorem 2.1. Now Theorem 2.1 implies that $F^*(G) = O_p(G) \neq 1$ and that $\chi = \theta^G$, where $\theta \in \mathrm{Irr}(H)$ for some p-subgroup H of G. Since H is monomial, χ is monomial. ∎

We remark that the proof of the above-mentioned theorem of Kimmerle, Lyons, Sandling and Teague uses the classification of the finite simple groups.

We also note the following characterization:

PROPOSITION 2.2. *Let G be a finite group, π a set of primes and $\chi \in \mathrm{Irr}(G)$. Then $|G|/\chi(1)$ is a π-number if and only if χ vanishes on all π'-elements of G.*

PROOF. If $|G|/\chi(1)$ is a π-number, then χ has defect zero for each prime $q \in \pi'$ and so $\chi(g) = 0$ for all q-singular elements. Thus χ vanishes on all π'-elements. Conversely, suppose that χ vanishes on all π'-elements. Let $q \in \pi' \cap \pi(G)$ and $Q \in \mathrm{Syl}_q(G)$. Then χ_Q vanishes on $Q - \{1\}$ and so χ_Q is a multiple of the regular character of Q whose degree is $|Q|$. It follows that $|Q| \mid \chi(1)$ and so q does not divide $|G|/\chi(1)$ for all $q \in \pi' \cap \pi(G)$, as desired. ∎

PROOF OF THEOREM 5. Let G be a counterexample of minimal order. The following fact about a group X will be used:

(*) If $O_p(X) \neq 1$ for some prime q, then $q \mid |X|/\rho(1)$ for all $\rho \in \mathrm{Irr}(X)$.

Fact (*) follows from Clifford's theorem, namely that $|X|/\rho(1) = (|X:O_q(X)|/l) \times (|O_q(X)|/\lambda(1))$, for some $\lambda \in \mathrm{Irr}(O_q(X))$ and l a divisor of $|X:O_p(X)|$. As $q \mid |O_q(X)|/\lambda(1)$, (*) follows.

Case I. Assume that G is nonsolvable. Suppose first that $F(G) \neq 1$. Let $p \in \pi(F(G))$ and set $M = O_p(G)$. By induction G/M is isomorphic to one of the following groups: A_5, PSL(2.7), A_6, PSL(2,8), PSL(2,17), S_6, M_{10}, PΓL(2,8). Note that G/M has a normal subgroup N/M which is isomorphic to one of the simple groups A_5, A_6, PSL(2,7), PSL(2,8), PSL(2,17). Then $\pi(N/M) = \{r,s,t\}$ and from [6] we see that there are $\theta_1, \theta_2, \theta_3 \in \mathrm{Irr}(N/M) - \{1_{N/M}\}$ with $|N/M|/\theta_1(1)$ an $\{r,s\}$-number, $|N/M|/\theta_2(1)$ an $\{r,t\}$-number and $|N/M|/\theta_3(1)$ an $\{s,t\}$-number. As $M = O_p(N)$, Fact (*) implies that $p \mid |N|/\theta(1)$ for all $\theta \in \mathrm{Irr}(N/M)$. It follows that $p \in \{r,s\} \cap \{r,t\} \cap \{s,t\}$. This is a contradiction. It follows that $F(G) = 1$ and so $F^*(G)$ is a direct product of simple groups. If $G > F^*(G)$, then induction implies that $F^*(G)$ is isomorphic to one of the following groups: A_5, A_6, PSL(2,7), PSL(2,8), PSL(2,17). Since $G \leq \mathrm{Aut}(F^*(G))$, we are left with only few possibilities for G and we use [6] to conclude that $G \cong A_5$, PSL(2,7), A_6, PSL(2,8), PSL(2,17), S_6, M_{10}, PΓL(2,8). This is a contradiction, as G is a counterexample. Thus $G = F^*(G)$ is a direct product of simple groups. If G is not simple, write $G = S \times R$, with S simple and $|\pi(R)| > 2$. Let $\tau \in \mathrm{Irr}(S) - \{1_S\}$, view τ as a character of G and conclude that $|R| \mid |G|/\tau(1)$, a contradiction. Thus G is simple and we get a contradiction from [4].

Case II. G is solvable. By the choice of $G, |\pi(G)| > 2$. Suppose that $|\pi(F(G))| \geq 2$ and let $p, r \in \pi(F(G))$. By Fact (*), $pr \mid |G|/\alpha(1)$ for all $\alpha \in \mathrm{Irr}(G)$. By our assumption, $\Sigma(G) = \{p, r\} < \pi(G)$. Now Theorem 1 implies that G satisfies conclusion (2)(b) of our theorem. This contradicts the choice of G. Therefore $F = F(G) = O_p(G)$ for some prime p. Also by the choice of $G, G/F$ is nonabelian.

Set $K = \Phi(G)$. Then $F(G/K) = F(G)/K$ is a completely reducible $G/F(G)$ module (see [10], III.4.5, p. 279). Assume that $K \neq 1$. By induction, G/K satisfies conclusion (2) of the theorem. If G/K satisfies either (2)(a), (2)(c) or (2)(d), so does G, a contradiction. Hence G/K satisfies (2)(b). Then $G'K/K$ is a nilpotent $\{p,r\}$-group for some prime r and $pr \mid |G'K/K|$. We get that G' is a nilpotent $\{p,r\}$-group as well. Hence, $G' \leq F(G) = O_p(G)$ so that G/F is abelian, a contradiction. We conclude that $K = 1$ so that F is an elementary abelian faithful completely reducible $G/F(G)$ module.

Let $q \in \pi(F(G/F))$. Then $q \neq p$ and $q \mid |G/F|/\theta(1) \mid |G|/\theta(1)$ for all $\theta \in \mathrm{Irr}(G/F)$. Then $pq \mid |G|/\theta(1)$ for such θ's. Since there exists a nonlinear $\theta \in \mathrm{Irr}(G/F)$ we get that $F(G/F) = O_q(G/F)$ and that $q \in \Sigma(G/F) \leq \{p,q\} < \pi(G/F)$. By Theorem 1, $(G/F)'$ is nilpotent so that $(G/F)' \subseteq F(G/F)$ is a q-group. Set $\bar{G} = G/F$, then $\bar{G}' \leq O_q(\bar{G})$. From Theorem 1 we also know that $\bar{G}$ has a Hall $\{p,q\}'$-subgroup $\bar{T} \neq 1$ such that $\bar{T}$ acts fixed-point-freely on $\bar{G}'$.

Set $\bar{H} = \bar{G}'\bar{T}$. Then $\bar{H}$ is a normal subgroup of $\bar{G}$, which is a Frobenius group with Frobenius kernel $\bar{G}'$ and a complement $\bar{T}$. Now, $\bar{H}$ acts faithfully, completely reducibly and coprimely on $F(G)$. Write $F(G) = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where the V_i are irreducible $\bar{H}$-modules. Since $\bar{G}' \neq 1$, there exists $V \in \{V_1, V_2, \ldots, V_m\}$ on which $\bar{G}'$ acts nontrivially. Then $\bar{C} := C_{\bar{H}}(V) < \bar{G}'$ (by [10], V, Satz 8.16). It is well known that $\bar{H}/\bar{C}$ is also a Frobenius group with Frobenius kernel $\bar{G}'/\bar{C} \neq 1$ (as $\bar{C} \leq \bar{G}'$) and Frobenius complement $\bar{T}\bar{C}/\bar{C} \cong \bar{T} \neq 1$ (as $\bar{T} \cap \bar{C} = 1$). By ([11], Th. 15.6) we get that $\dim(V) = |\bar{T}|\dim(C_V(\bar{T}))$. In particular, $\dim(C_V(\bar{T})) \neq 0$. Choose $x \in C_V(\bar{T}) - \{1\}$. Then $\bar{T}\bar{C} \leq C_{\bar{H}}(x)$.

We now apply character correspondence to the coprime action of $\bar{H}$ on V ([11], Th. 13.24). Since $\bar{T}\bar{C}$ fixes $x \neq 1$, we know that $\bar{T}\bar{C}$ fixes some $\lambda \in \mathrm{Irr}(V) - \{1_V\}$. Let $I_{\bar{H}}(\lambda)$ be the inertia group of λ in $\bar{H}$, then $\bar{T}\bar{C} \leq I_{\bar{H}}(\lambda)$. Clearly, we can consider λ as a character of $V_1 \oplus V_2 \oplus \cdots \oplus V_m \cong F(G)$. Set $\bar{H} = H/F(G)$, and $\bar{T} = T/F(G)$ for H and T subgroups of G. Let I be the inertia group of λ in H. We get that $T \leq I$.

As H acts coprimely on $F(G)$, we can extend λ to $\lambda^* \in \mathrm{Irr}(I)$ (see, e.g., [11], Cor. 6.27). Set $\mu = (\lambda^*)^H$. Then $\mu \in \mathrm{Irr}(H)$. Now, $\mu(1) = \lambda(1)|H:I| = |H:I|$. If $I = H$, then $\bar{H} = I_{\bar{H}}(\lambda)$ and again by character correspondence $\bar{H}$ fixes some $y \in V - \{1\}$. But then the irreducible $\bar{H}$-module V would contain the trivial module implying that V is the trivial module. Thus $H \neq I$. As $T \leq I$ and $\bar{G}' \leq O_q(\bar{G})$ we get that $\mu(1) = q^s$ for some natural number $s > 0$, and $|T| \mid |I| = |H|/\mu(1)$. Recall that H is normal in G. Hence, the co-degrees of all nonlinear irreducible characters of H have at most two distinct prime divisors. As $F(G)$ is a p-group and $\bar{T}$ a $\{p,q\}'$-group we conclude that $\bar{T}$ is an r-group for a prime $r \in \{p,q\}'$. But $\bar{T}$ is a Hall $\{p,q\}'$-subgroup of $\bar{G}$. Therefore $\pi(\bar{G}) \subseteq \{p,q,r\}$ so that $\pi(G) = \{p,q,r\}$. ■

We now give an example showing that if $G/F(G)$ is abelian in Theorem 5 (part 2d) then $\Sigma(G)$ is unbounded in general. The example also shows that analogs for [8] and [9] are false for co-degrees. See the remark after the statement of Theorem 5 in the introduction.

EXAMPLE. Let $r_1, r_2, \ldots, r_n$ be distinct primes and C_i the cyclic group of order r_i, $i = 1,2,\ldots,n$. Set $C = C_1 \times C_2 \times \cdots \times C_n$. Let p be a prime, $p \notin \{r_1, r_2, \ldots, r_n\}$. For each $i = 1,2,\ldots,n$, we choose a $\mathrm{GF}(p)[C]$-module M_i such that C/C_i acts fixed-point-freely on M_i. Let $M = M_1 \times M_2 \times \cdots \times M_n$ and $G = MC$. Here M is normal in G.

Let $\chi \in \mathrm{Irr}(G)$, $\chi(1) \neq 1$. Then $\chi_M = e(\lambda_1 + \lambda_2 + \cdots + \lambda_t)$, where $\lambda_1, \lambda_2, \ldots, \lambda_t$ are all the conjugates in G of $\lambda = \lambda_1 \in \mathrm{Irr}(M)$. Let $I(\lambda)$ be the inertia group of λ in G, then $t = |G:I(\lambda)|$. Since $e^2 \mid |G/M| = |C|$ and as $|C|$ is square-free, we get that $e = 1$. It follows that $\chi(1) = t\lambda(1) = t$. Note that $\lambda \neq 1_M$. Write $\lambda = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ where $\mu_i \in \mathrm{Irr}(M_i)$. As M_i is normal in G for all $i = 1,2,\ldots,n$, we get that $I_C(\lambda) = \bigcap_{i=1}^n I_C(\mu_i)$ (here $I_C(\lambda)$ and $I_C(\mu_i)$ are the corresponding inertia groups). The cyclic group C/C_i acts fixed-point-freely on M_i and therefore it acts fixed-point-freely on $\mathrm{Irr}(M_i)$. This means that if $\mu_i \neq 1_{M_i}$, then $I_C(\mu_i) = C_i$.

As $\lambda \neq 1$, there exists an index i such that $\mu_i \neq 1_{M_i}$. If $\mu_j = 1_{M_j}$ for all $j \neq i$ then $I_C(\lambda) = C_i$. If $\mu_j \neq 1_{M_j}$ for some $j \neq i$ then $I_C(\lambda) \leq I_C(\mu_i) \cap I_C(\mu_j) \subseteq C_i \cap C_j = 1$. Now, $I(\lambda) = MI_C(\lambda)$ and hence $\chi(1) = |G:I(\lambda)| = |C:I_C(\lambda)| = \prod_{k=1}^n r_k$ or

$\Pi_{k=1,k\neq i}^{n} r_k$ for some i. Consequently, $|G|/\chi(1) = |M||C|/t = |M|$ or $|M|r_i$ for some i.

Note that if we take a nonprincipal irreducible character of M_i, consider it as a character of M, induce it to G and take an irreducible constituent Θ of this induced character, then the above shows that the co-degree of Θ is $|M|r_i$. We see now that the co-degrees of all nonlinear irreducible characters of G has at most two distinct prime divisors. However, $\Sigma(G) = \{p, r_1, r_2, \ldots, r_n\}$ is not bounded.

We now prove:

THEOREM 2.3. *Let G be a finite group. Assume that for every nonlinear $\chi \in \mathrm{Irr}(G)$, the number $|G|/\chi(1)$ is cube-free. Then one of the following holds*:

(1) *G is a solvable group of derived length at most* 3. *If G has odd order then the derived length is at most* 2.

(2) *$G = S \times C$ where C is a group of odd order with derived length at most* 2 *and $S \cong \mathrm{PSL}(2,p)$ with p a prime, $p \equiv 3$ or 5 (mod 8).*

PROOF. Let G be a counterexample of minimal order. Note that the assumption of the theorem is inherited by normal subgroups and quotient groups.

Let $p \in \pi(G)$ and set $|G|_p = p^a$ for a natural number a. Assume that $a \geq 3$. Since $p^3 \nmid |G|/\chi(1)$ for all nonlinear $\chi \in \mathrm{Irr}(G)$ we get that $p^{a-2} | \chi(1)$ for all such χ. Thus $|G| = |G:G'| + kp^{2(a-2)}$ for some natural number k. If $2(a-2) \geq a$, then $p^a \,|\, |G:G'|$ and so $(p, |G'|) = 1$. If $2(a-2) < a$, then $a = 3$ and $p^2 \,|\, |G:G'|$ so $|G'|_p | p$. Hence

(*) For every $p \in \pi(G')$, either $|G|_p \leq p^2$ or $|G|_p = p^3$ and $|G'|_p = p$.

Case 1. Assume that G is solvable. Let N be a minimal normal subgroup of G. If G has another minimal normal subgroup M we imbed G into $(G/N) \times (G/M)$ and use induction to get a contradiction. Hence N is the unique minimal normal subgroup. In particular $F = F(G) = O_p(G)$ for some prime p. Note that as $N \subseteq G'$, $p \in \pi(G')$. Assume that $|F| \geq p^3$. If G/F is nonabelian we take a nonlinear $\rho \in \mathrm{Irr}(G/F)$. Then $|F| \,|\, |G|/\rho(1)$, a contradiction. Hence, G/F is abelian and so $G' \subseteq F$. Now (*) shows that $|G'| = p$ so that $G'' = 1$, a contradiction. Therefore $|F| = p$ or p^2. Now $C_G(F) = F$. If F is cyclic, G is metabelian, a contradiction. Thus F is elementary abelian of order p^2.

Assume that $N < F$. Then $|N| = p$. Set $U = C_G(N) \cap C_G(F/N)$. Note that U stabilizes the series $1 < N < F$ and so U is a p-group. It follows that $U \leq F$. Let $S \in \mathrm{Syl}_p(G)$ be such that $F \leq S$. Then $N \leq Z(S)$ and $F/N \leq Z(S/N)$. Thus $S \leq U \leq F$ and so $F = S$. So $G = FK$ with K a Hall p'-subgroup of G which leaves N

invariant. But then K has to leave another subgroup of F invariant contrary to the fact that N is a unique minimal normal subgroup. We conclude that $N = F$. In particular $F \leq G'$. By $(*)$ we get that $|G|_p = p^2$ so $G/F(G)$ is a p'-group isomorphic to a solvable irreducible subgroup of $\mathrm{GL}(2,p)$. We get a contradiction by inspecting ([17], Chapter II). The inspection could be done as follows: If G/F is induced or semi-linear, then the conclusion is easy. In the remaining cases we have $F(G/F) = QT$ with $T = Z(G/F)$ a cyclic group and Q isomorphic to Q_8. Also $T \cap Q = Z(Q)$ and $(G/F)/(Z(G/F)) \cong S_3$ or Z_3. Now the theorem is proved for solvable groups.

Case 2. Assume that G is nonsolvable. Set $L = G^{(\infty)}$. If $L < G$, then by induction $L \cong \mathrm{PSL}(2,r)$ with r a prime such that $r \equiv 3$ or $5 \pmod 8$. Since $|\mathrm{Out}(L)| = 2$ we get that $L \times C_G(L)$ has index 1 or 2 in G. If the index is 2 then $G/C_G(L) \cong \mathrm{PGL}(2,r)$ which has a character degree equal to r, so the corresponding co-degree is divisible by 8, a contradiction. Hence $G = L \times C_G(L)$. Since L has a co-degree divisible by 4, $C_G(L)$ has to be of odd order. Note that $C_G(L)$ also satisfies the assumption of the theorem so that $C'' = 1$ by the solvable case. This is a contradiction as G is a counterexample. It follows that $G = G'$. By $(*)$ we know that $|G|$ is cube-free. Let N be a minimal normal subgroup of G. If $N = G$ then, by [18], $G \cong \mathrm{PSL}(2,r^m)$ with r^m a prime power such that $r^m \equiv 3$ or $5 \pmod 8$. As $|G|$ is cube-free $m = 1$ or 2, but if $m = 2$, $r^2 \equiv 1 \pmod 8$. Thus $m = 1$. So $G \cong \mathrm{PSL}(2,r)$, a contradiction. Thus $N < G$. If N is nonabelian, then $|N|_2 = 4$ and as $|G|_2 = 4$, $(G/N)' < G/N$ forcing $G' < G$, a contradiction. It follows that N is elementary abelian of order q or q^2 for some prime q. By induction, $G/N \cong \mathrm{PSL}(2,r)$ with r a prime such that $r \equiv 3$ or $5 \pmod 8$. If $|N| = q$, $N < Z(G)$. Now, the Schur multiplier of G/N has order 2 ([10], V.25.7, p. 646) and G is perfect, so $G \cong \mathrm{SL}(2,r)$. But then $|G|$ is divisible by 8, a contradiction. We conclude that $|N| = q^2$. Hence $N \in \mathrm{Syl}_q(G)$ and therefore $G = SN$, $S \cong G/N$, $(|S|,|N|) = 1$. Since S is faithful on N, S is isomorphic to a subgroup of $\mathrm{SL}(2,q)$. This is impossible as $\mathrm{SL}(2,q)$ has one involution while S has more than one. ∎

Remark. The bound on the derived length for solvable groups in Theorem 2.3 is best possible, as the symmetric group on four letters shows.

Remarks on Examples. We can obtain some examples of characters of prime power co-degrees as follows. A group G is termed a *Camina group*, if it has a normal subgroup N, such that for each element $g \notin G - N$, we have that $|C_G(g)| = |C_{G/N}(gN)|$. Equivalently, every irreducible character not containing N in its kernel vanishes outside N. This includes Frobenius groups with N the Frobenius kernel. It is shown in [3] that if G is not Frobenius with N the kernel, then either G/N or N is a p-group for some prime p. Suppose that G is a Camina group with N a p-group. Then each character of G not containing N in its kernel vanishes on all p'-elements. By Proposition 2.2, all the co-degrees of these characters are p-powers. It also follows that the structure of groups possessing such characters can be rather involved (see, e.g., [5]).

Obviously, this example can be generalized somewhat, by looking at groups G, containing a normal p-subgroup N such that for all p'-elements $g \in G$ we have that $|C_G(g)| = |C_{G/N}(gN)|$. Further, we can replace p by a set of primes π. This wider class can be characterized in terms of characters with co-degrees which are π-numbers as follows:

PROPOSITION 2.4. *Let G be a finite group and π a proper subset of $\pi(G)$. Let N be the intersection of all the kernels of the irreducible characters of G which do not have π-numbers as co-degrees. Assume that N is not trivial. Then N is a nilpotent normal π-subgroup, such that for each π'-element g, the equality $|C_G(g)| = |C_{G/N}(gN)|$ holds, and g acts on N without fixed points.*

PROOF. By Proposition 2.2, any character that does not contain N in its kernel vanishes on π'-elements. This yields the equality of orders of the centralizers. Let $q \notin \pi$ be a prime and let g be a nonidentity element in the center of a Sylow q-subgroup Q of G. Then $|Q| \mid |C_G(g)| = |C_{G/N}(gN)|$ and, in particular, $|G|_q \mid |G/N|$ for all $q \in \pi(G) - \pi$. This shows that N is a π-group.

Let g be a π'-element of prime order and let $p \in \pi$. Take P/N a Sylow p-subgroup of $C_{G/N}(gN)$. Then $P = NC_P(g) = NR$, where R is any Sylow p-subgroup of $C_P(g)$. Note that $|P/N| = |C_{G/N}(gN)|_p = |C_G(g)|_p = |R|$. Thus, $N \cap R = 1$. But $N \cap R \in \mathrm{Syl}_p(C_N(g))$. Hence g acts on N without fixed points. In particular N is nilpotent. ∎

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